The Application of Signal Detection Theory to Optics

PROGRESS REPORT

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ABSTRACT

The mutual information between the radiance of an object plane and the aperture field of an observing optical instrument is calculated under a threshold approximation. The mutual information depends on the size $\lambda R/a$ of the conventional resolution element, where λ is the wavelength of the light, R is the distance of the object, and a is the diameter of the aperture. Details smaller than $\lambda R/a$ do not contribute significantly to the mutual information.

Optical Information Theory

How much information about an object plane exists in the image formed by an optical instrument when the light from the object is received in the presence of background light? This problem has mostly been treated under the assumption that the object, the image, and the background--or "noise"--can be described as Gaussian random processes--an assumption generally recognized as hazardous when the object radiates incoherently and the image is represented by a distribution of illuminance.[1-10]

The maximum information that an optical instrument can obtain about an object plane in its field of view resides in the values of the electromagnetic field at its aperture, for these are the data that it processes by means of lenses, stops, photosensitive surfaces, and so on, in order to form and record images, on the basis of which conclusions are drawn about the object. When the object light and the background are strong enough to be treated classically, the field values at the aperture are samples of a spatio-temporal Gaussian random process. They are described by joint probability density functions (pdf's) that depend only on the radiance distribution of the object plane and the spectral density of the background light.

The system "object plane + propagation medium + optical instrument" can be treated as a spatio-temporal communication channel in which the transmitted symbols are patterns of radiance $B_m(\underset{\sim}{u})$ in the object plane, and the received data are the values of the electromagnetic field at the aperture. Assuming the validity of a scalar theory of light, the field at the aperture is represented by the analytic signal $\psi_+(\underset{\sim}{r}, t)$, the positive-frequency part of the field at point $\underset{\sim}{r} \in A$ and at time t. We can imagine a finite set of L different patterns

$B_m(\underset{\sim}{u})$, m = 1, 2, ..., L, each corresponding to a different symbol of an alphabet into which messages are coded, a new pattern being exposed every T seconds. The light emitted by each pattern will be assumed to be quasimonochromatic and to have the same spectral composition, represented by its spectral density $X(\omega)$. In $X(\omega)$ the angular frequency ω is referred to the central carrier frequency $\Omega = kc = 2\pi c/\lambda$, where λ is the wavelength of the object light.

For ordinary sources of incoherent light it will be possible to change the radiance patterns $B_m(\underset{\sim}{u})$ only at intervals T much greater than the reciprocal bandwidth W^{-1} of the light; WT >>1. For example, for a source of spectral width 100 Å.U. at a wavelength 5150 Å.U., $W^{-1} \cong 10^{-13}$sec. With the transmission of each pattern, therefore, is associated a large number WT of independent temporal degrees of freedom in the aperture field. The energy radiated by the object for the duration T of each pattern is divided among these degrees of freedom, and each receives only a small fraction. As a result, this incoherent optical communication system can to good approximation be treated as a "very noisy" channel.[11]

The specific approximation that is being made is that E/NWT <<1, where E is the total energy received at the aperture from the object during the time T, and N is the spatio-temporal spectral density of the background, assumed white. We can set N equal to $K\mathcal{T}$, where K is Boltzmann's constant and $\mathcal{T}$ is the effective absolute temperature of the background. The quantity E/AWT, where A is the area of the aperture, is equal to the illuminance per Hz at the aperture. As an example, the value of E/AWT for moonlight at a point near the ground has been measured as $1.42 \cdot 10^{-18}$ watts/m^2 Hz at 5150 Å.U.[12] For an effective temperature 300°K and an aperture of area A = 1 cm^2, the ratio E/NWT equals 0.034.

When $E/NWT \ll 1$, it is possible to use the threshold approximation for the likelihood ratio formed by dividing the joint p.d.f. of the field values $\psi_+(\underset{\sim}{r}, t)$ for a radiance pattern $B_m(\underset{\sim}{u})$ by their joint p.d.f. when the object does not radiate.[13] Applying the formula given by Gallager,[11] one finds for the average mutual information $I(B; \psi)$ per pattern between the transmitted patterns $B_m(\underset{\sim}{u})$ and the observed field values $\psi_+(\underset{\sim}{r}, t)$,

$$I(B; \psi) = (\overline{E}^2/2N^2WT)\ \overline{B}_T^{-2}\ V \quad \text{Nats}, \tag{1.1}$$

where $\overline{E}$ is the average total energy received from the patterns $B_m(\underset{\sim}{u})$ during T seconds and is proportional to

$$\overline{B}_T = \int_0 \langle B_m(\underset{\sim}{u})\rangle_m\ d^2\underset{\sim}{u}, \tag{1.2}$$

the average total radiance of the object plane (0 indicates an integration over the object plane). Here $\langle\cdot\rangle_m$ denotes an average with respect to the set of probabilities q_m with which the patterns $B_m(\underset{\sim}{u})$ are used in transmitting messages,

$$\langle B_m(\underset{\sim}{u})\rangle_m = \sum_{m=1}^{L} q_m\ B_m(\underset{\sim}{u}). \tag{1.3}$$

The quantity V in Eq. (1.1) is given by

$$V = \int_0 \int_0 \Psi_B(\underset{\sim}{u}_1, \underset{\sim}{u}_2)\ |\mathcal{g}(\underset{\sim}{u}_1 - \underset{\sim}{u}_2)|^2\ d^2\underset{\sim}{u}_1 d^2\underset{\sim}{u}_2 \tag{1.4}$$

where the function

$$\Psi_B(\underset{\sim}{u}_1, \underset{\sim}{u}_2) = \langle \delta B_m(\underset{\sim}{u}_1)\ \delta B_m(\underset{\sim}{u}_2)\rangle_m, \tag{1.5}$$

with $\delta B_m(\underset{\sim}{u}) = B_m(\underset{\sim}{u}) - \langle B_m(\underset{\sim}{u})\rangle_m$, is the autocovariance of the radiance patterns in an ensemble governed by the probabilities q_m, and where the kernel

$$\mathcal{G}(\underset{\sim}{u}) = A^{-1} \int_A \exp(ik\underset{\sim}{u} \cdot \underset{\sim}{r}/R)d^2\underset{\sim}{r}, \tag{1.6}$$

with $k = 2\pi/\lambda$ and R the distance of the object, is the Fourier transform of the indicator function of the aperture.[14]

The autocovariance function $\Psi_B(\underset{\sim}{u}_1, \underset{\sim}{u}_2)$ describes the average structure of the patterns $B_m(\underset{\sim}{u})$. If we suppose that there are a great many of them and treat them as drawn from an ensemble of two-dimensional homogeneous random processes, the autocovariance $\Psi_B(\underset{\sim}{u}_1, \underset{\sim}{u}_2)$ will be a function only of $\underset{\sim}{u}_1 - \underset{\sim}{u}_2$ that is the narrower, the finer the structure of the patterns. Eq. (1.4) shows, however, that there is no point to putting details into the patterns that are smaller than the width of the kernel $|\mathcal{G}(\underset{\sim}{u}_1 - \underset{\sim}{u}_2)|^2$, which is of the order of $\lambda R/a$, where a is the diameter of the aperture. Indeed, Schwarz's inequality shows that V is maximum when $\Psi_B(\underset{\sim}{u}_1, \underset{\sim}{u}_2)$ is proportional to $|\mathcal{G}(\underset{\sim}{u}_1 - \underset{\sim}{u}_2)|^2$,

$$\Psi_B(\underset{\sim}{u}_1, \underset{\sim}{u}_2) = (\text{Var } B_m) \; |\mathcal{G}(u_1 - u_2)|^2, \tag{1.7}$$

where $\text{Var } B_m = \Psi_B(\underset{\sim}{u}, \underset{\sim}{u})$ is the variance of the set of random patterns when used with probabilities q_m. Hence the mutual information is bounded by

$$I(B;\ \psi) \le I_{max} =$$

$$(\overline{E}^2/2N^2WT)\ (\text{Var } B_m/\overline{B}_T{}^2) \int_0 \int_0 |\mathcal{G}(u_1 - u_2)|^2\ d^2\underset{\sim}{u}_1 d^2\underset{\sim}{u}_2$$

$$= \frac{\overline{E}}{2N^2MWT} \quad \frac{\text{Var } B_m}{\overline{B}^2} \left(1 - \frac{16}{3\pi^2}\right), \tag{1.8}$$

the latter expression holding for a circular aperture, with $\overline{B} = \overline{B}_T/A_o$, A_o the area of the patterns, and

$$M = A\, A_o/\lambda^2 R^2 \tag{1.9}$$

the number of spatial degrees of freedom in the aperture field. For a rectangular aperture the numerical factor is 2/9 rather than $\frac{1}{2}\left(1 - \frac{16}{3\pi^2}\right) = 0.2298$.

The mutual information can be made as large as desired by using patterns with a large variance. In practice, however, the radiance $B_m(\underset{\sim}{u})$ will be restricted to a finite range. The ratio $(\text{Var } B_m)/\overline{B}^2$ is then greatest if $B_m(\underset{\sim}{u})$ equals 0 or $2\overline{B}$ with equal probability. The patterns $B_m(\underset{\sim}{u})$ should therefore be chosen so as to have maximum contrast and an autocovariance given by Eq. (1.7).

If the patterns are generated by point sources located at points $\underset{\sim}{u}_p$ in the object plane, we can put

$$B_m(\underset{\sim}{u}) = \sum_p b_{mp}\, \delta(\underset{\sim}{u} - \underset{\sim}{u}_p), \tag{1.10}$$

where b_{mp} measures the strength of the p-th source when the m-th pattern is being displayed. The mean total radiance is then

$$\overline{B}_T = \sum_p \langle b_{mp} \rangle_m \int \delta(\underset{\sim}{u} - \underset{\sim}{u}_p)\, d^2\underset{\sim}{u}$$

$$= \sum_p \langle b_{mp} \rangle_m = M_s\, \overline{b} \tag{1.11}$$

if the average strength $\langle b_{mp} \rangle_m$ of each source is the same and equal to $\bar{b}$, and if there are M_s sources. The "autocovariance" is now

$$\Psi_B(\underset{\sim}{u}_1, \underset{\sim}{u}_2) = \sum_p \sum_q \langle (b_{mp} - \bar{b})(b_{mq} - \bar{b}) \rangle_m \delta(\underset{\sim}{u}_1 - \underset{\sim}{u}_p)\ \delta(\underset{\sim}{u}_2 - \underset{\sim}{u}_q), \tag{1.12}$$

and the mutual information is, from Eqs. (1.1) and (1.4),

$$I(B; \psi) = (\bar{E}^2/2N^2M_s{}^2WT)\ \bar{b}^{-2} \sum_{p,q} \beta_{pq}\ |\mathcal{G}(\underset{\sim}{u}_p - \underset{\sim}{u}_q)|^2 \tag{1.13}$$

where

$$\beta_{pq} = \langle (b_{mp} - \bar{b})(b_{mq} - \bar{b}) \rangle_m \tag{1.14}$$

is the autocovariance of the source strengths.

For a rectangular aperture $a_x \times a_y$,

$$\mathcal{G}(\underset{\sim}{u}) = \mathrm{sinc}(u_x/\delta_x)\ \mathrm{sinc}(u_y/\delta_y), \tag{1.15}$$

where $\mathrm{sinc}\ x = (\sin \pi x)/\pi x$ and

$$\delta_x = \lambda R/a_x, \qquad \delta_y = \lambda R/a_y \tag{1.16}$$

are the resolution elements in the x and y directions. By placing the sources at points separated in x by δ_x and in y by δ_y, we obtain a mutual information

$$\begin{aligned} I(B; \psi) &= (\bar{E}^2/2N^2M_s{}^2WT)\ \bar{b}^{-2} \sum_p \beta_{pp} \\ &= (\bar{E}^2/2N^2MWT)\ (\mathrm{Var}\ b_{mp})/\bar{b}^2, \end{aligned} \tag{1.17}$$

since $|\mathcal{G}(\underset{\sim}{u}_p - \underset{\sim}{u}_q)|^2 = \delta_{pq}$. Here we have assumed that the variances $\mathrm{Var}\ b_{mp} = \beta_{pp}$ of all source strengths are equal. Now $M_s = A_o/\delta_x\ \delta_y = A\ A_o/\lambda^2\ R^2$ equals M, the number of spatial degrees of freedom in the aperture field. The ratio ($\mathrm{Var}\ b_{mp}/\bar{b}^2$) is maximum under a limitation on source power when $b_{mp} = 0$ and $b_{mp} = 1$ with equal probability. The mutual information is then

$$I(B;\ \psi) = \bar{E}^2/2N^2MWT, \tag{1.18}$$

which is larger than that obtained for random patterns.

With M sources that are turned either on or off, the number of possible patterns is $L = 2^M$, and the maximum attainable mutual information with a perfect optical system and no background would be $M\ \ln 2$. The approximations that entered the calculation of Eq. (1.18) and its predecessors would appear, therefore,to be invalid unless $(\bar{E}/M)^2/N^2WT << 1$, a rather more stringent condition than the condition $\bar{E}/NWT$ permitting use of the threshold expansion of the likelihood ratio.

The basic result of this work is Eqs. (1.1) and (1.4), which combined give the mutual information between aperture field and object radiance as

$$I(B;\ \psi) = (\bar{E}^2/2N^2WT)\ \bar{B}_T^{-2} \times \int_0 \int_0 \Psi_B(\underset{\sim}{u}_1,\ \underset{\sim}{u}_2)\ |\mathcal{G}(\underset{\sim}{u}_1 - \underset{\sim}{u}_2)|^2\ d^2\underset{\sim}{u}_1 d^2\underset{\sim}{u}_2, \tag{1.19}$$

a formula indicating that making the structure of the object radiance finer and finer does not increase the transmitted information indefinitely. A limit is set by the powers of resolution embodied in the observing optical instrument, which appear in Eq. (1.19) in the kernel $|\mathcal{G}(u_1 - u_2)|^2$, whose width is of the order of a conventional resolution element $\lambda R/a$. Structural details smaller than $\lambda R/a$ provide no significant amount of information.

References

1. A. Blanc-Lapierre, M. Perrot, G. Péri, Optica Acta 2, 1-5 (April, 1955).
2. G. Toraldo di Francia, Optica Acta 2, 5-8 (April, 1955).
3. E. H. Linfoot, J. Opt. Soc. Am. 45, 808-819 (October, 1955).
4. P. B. Fellgett and E. H. Linfoot, Phil. Trans. Roy. Soc. (London), 247, no. A. 931, 369-407 (February 17, 1955).
5. E. H. Linfoot, Optica Acta 5, 1-14 (March-June, 1958).
6. E. H. Linfoot, Physica 24, 476-494 (1958).
7. R. C. Jones, J. Opt. Soc. Am. 50, 1166-1170 (December, 1960).
8. R. C. Jones, J. Opt. Soc. Am. 51, 1159-1171 (November, 1961).
9. R. C. Jones, J. Opt. Soc. Am. 52, 1193-1200 (November, 1962).
10. B. R. Frieden, J. Opt. Soc. Am. 58, 1105-1112 (August, 1968).
11. R. G. Gallager, Information Theory and Reliable Communication, J. Wiley & Sons, Inc. (1968), pp. 147-149.
12. A. Boileau, Visibility Laboratory, Scripps Institution of Oceanography (private communication).
13. C. W. Helstrom, J. Opt. Soc. Am. 59, 164-175 (February, 1969).
14. Ref. 13, Eq. (A4).